Different Meanings of Pure Imaginary Numbers

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Abstract: This study investigated in-service teachers' conceptualization of the pure imaginary number ib, within the Cartesian form, $a + ib$ where *a* and *b* are real numbers and i is the imaginary unit. As part of a larger design-based research study, in which a professional development (PD) program was designed to investigate five in-service teachers' conceptualization of different forms of complex numbers, we conducted pre and post written problem-solving sessions together with post PD interviews. Results showed that after PD all the participants defined i as one of the roots of the quadratic equation, $x^2 + 1 = 0$. They also could show i geometrically as a point (0,1) on the complex plane. Although all the participants mentioned operator meaning of i as a 90-degree rotation when multiplied with b; only one of them mentioned the dilation meaning of b when multiplied with i. Results suggested considering the pure imaginary part of the Cartesian form focusing on both the operator meanings of b and i is important for understanding the nature of complex numbers and the complex plane for teacher education and teacher content knowledge. These results further suggest that quantitative reasoning might lay a foundation for connecting different forms of complex numbers, including the unit, i.

Keywords: Complex numbers, teacher knowledge, quantitative reasoning.

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Introduction

At different stages of schooling, students are expected to work with and build relationships among number systems (CCSM, 2010). Understanding of complex numbers is specifically important in science, technology, engineering, and mathematics related fields (STEM). Complex numbers are particularly crucial in the learning of advanced mathematics topics such as algebra and complex analysis. They also play an important role in the learning of concepts of advanced physics such as quantum physics (Karam, 2020) and different fields of engineering including relativity, electromagnetic theory, signal processing (Atmaca, 2014), hydrodynamics and electrical circuits (Benitez et.al, 2012). Thus, a robust conception of complex numbers is expected of students both for their access to and success in STEM and other interdisciplinary fields (Anevska, Gokovska & Malcheski, 2015). Specifically, researchers stated that a complete conception of complex numbers necessitates geometrically thinking of a complex number as a point on the complex plane and as a vector (Fauconnier & Turner, 2002). Also, algebraically, "a complex number should be conceptualized as one number, i.e., the expression $a + ib$ is a single entity combining a real number and an imaginary number" (Nordlander & Nordlander, 2012, p. 633) such that they are mathematical objects in a well-defined set consisting of elements of the same kind or a particular category (Sfard, 1991). Yet, researchers argued that "Complex numbers …typically are instances of mathematical structures that seem to depend merely on formal arrangements in a system of meaningless signs, not referring to anything informal or quasi-empirical" (Glas, 1998, p. 367) such that students mostly "…cannot visualize them" and "…are also questioning the use of complex numbers in reality, and mention the difficulty in imagining what complex numbers 'stand for and really are'" (Nordlander & Nordlander 2012, p. 633). Researchers concluded that for students "…at least the imaginary unit must be visible" (p. 637) to think of any number as a complex number. Thus, they pointed to the understanding of the components of the Cartesian form meaningfully.

Specifically, Sfard (1991) further stated that the first stage for conceptualizing complex numbers is to recognize that $i = \sqrt{-1}$, in the Cartesian form. Regarding the root symbol, the radical sign $\sqrt{}$, Kontorovich (2018a) pointed that it "…initiates polysemy-a phenomenon in which the same concept or symbol can be interpreted in discrepant manners depending on the context in which they are used and on the curricular norms associated with the context" (p. 17). For example, $\sqrt{9}$ can be understood as equal to 3 since $y = \sqrt{x}$ is a function in Real numbers where for every x value in the domain there should be a single y value. Yet, in the field of complex numbers, $\sqrt{}$ is a multi-valued function such that $\sqrt{9}$ has two roots which are ±3 (Kontorovich, 2018b). In this respect, researchers argued that i should be understood as "…one of the square roots of (-1)" such that as a new discourse "…the expression $\sqrt{-1}$ is not a number no longer holds" (Nachlieli & Elbaum-Cohen, 2021, p. 5) this part needs to be developed for students and teachers.

Researchers also point to the understanding of i as a vector and a point (e.g., Karakok et al., 2015). Cognizing of

i as a vector with (0,1) pointwise representation is further related with understanding the ib component of the Cartesian form, a+ib, where b is any real number on the Real number line. That is, whenever one makes sense of ib as the multiplication of i with any real number, this can lead to their "…interpreting i as a rotation of the real line through 90°" (p. 967), which yields to the formation of complex plane (Harding & Engelbrecht, 2007).

Making sense of i as an operator is further important in two significant ways. Kontorovich et al. (2021) argued for two types of relations between number sets, namely a nested and a transition relation. In the nested relation, for example, Real numbers are considered as the subset of complex numbers. The transition view rather depends on an image that complex numbers are isomorphic to Real numbers where " two sets are isomorphic if there exists a bijection between their elements that preserves a binary relationship, for instance addition and multiplication" (p.262). They further pointed that isomorphic image might resolve the aforementioned issue of thinking of $\sqrt{9}$ as 3, approaching it as a real square root function, and thinking of $\sqrt{9}$ as an element of complex numbers yielding to ±3 (Kontorovich, 2018b). This is possible as thinking of the isomorphic image might lead to the understanding that "…identically appearing words and symbols can be interpreted rather differently in different number sets" (p. 263). Thus, making sense of i as an operator acting on any real number, say *b*, might further allow for the understanding of the ib component of $a + ib$ such that not only one makes sense of the fact that real numbers are a subset of complex numbers but also that there is an isomorphism between the two sets. This is also possible as isomorphic and nested images of sets are complementary viewpoints rather than conflicting ideas (Kontorovich et al., 2021). This is further important as researchers argued that "…it is useful for students and teachers to be able to flexibly switch between the two images" (Kontorovich et al., 2021, p. 264).

It is in this regard that, considering mathematics teachers as key figures in the preparation of students (NCTM, 2000), who need to have a good understanding of complex numbers for their future careers especially in STEM related fields, we scrutinized specifically in-service teachers' conceptualization of i both algebraically and geometrically. With the report in this paper, we attempt at extending previous research on complex numbers by further delineating different conceptualizations of i.

Conceptual Framework

Researchers pointed that a robust conception of complex numbers include one's knowing both the algebraic and geometric representations of the Cartesian, polar and the exponential forms (See Table 1), making sense of the connections among them and the flexibility to go among these forms (Karakok et.al., 2015)

However, research has shown both students' (Çelik & Özdemir, 2011; Nordlander & Nordlander, 2012; Panaoura et al., 2006; Soto-Johnson & Troup, 2014) and prospective and in-service mathematics teachers' difficulties on complex numbers, especially making connections among different representations (Conner et al.,

2007; Karakok et al., 2015; Nemirovsky et al., 2012). Therefore, researchers emphasized the simultaneous employment of algebraic and geometric aspects of complex numbers.

Table 1. Different representations of complex numbers (Karakok et.al., 2015, p.329)

Representation	Form			
	Purely imaginary i	Cartesian $(a + bi)$	Polar $r(\cos\theta + i\sin\theta)$	Exponential $re^{i\theta}$
Algebraic	$i, \sqrt{-1}, (0, 1)$	$a + bi, (a, b), z$	$r(\cos\theta + i\sin\theta), z$	$re^{i\theta}$
Geometric	A point on the complex plane, a rotation operator	A point on the complex plane, a vector with a magnitude of $\sqrt{a^2+b^2}$ and an angle of $\tan^{-1}\left(\frac{a}{b}\right)$ with the positive real axis, a rotation and dilation operator	A point on the circle centered at the origin with radius r, a vector with magnitude of r and an angle of θ with the positive real axis, a rotation and dilation operator	A vector with magnitude of r and an angle of θ with the positive real axis, a point on the circle centered at the origin with radius r, a rotation and dilation operator

Specifically, Soto-Johnson and Troup (2014) reported on mathematics major students' geometric and algebraic reasoning about complex-valued equations. Results of their study has shown that students were tended to think algebraically. Yet, when they were asked purposefully to think about geometric meaning of the equations, they could manage it. Therefore, researchers concluded that "requiring students to reason about a task both geometrically and algebraically appears to mesh these two types of reasoning" (p. 124). In addition, Nordlander and Nordlander (2012) study with engineering undergraduate and high school students showed that students "…have difficulties discerning the basic property of complex numbers, i.e., any number is a complex number" (Nordlander & Nordlander, 2012, p. 627). So, researchers concluded that for students "…at least the imaginary unit must be visible" (p. 637) to think of any number as a complex number. Similarly, Nachlieli and Elbaum-Cohen (2021) reported on a study with twelfth-grade students' mastery of complex numbers, which entailed acknowledging that "..the word number also signifies objects of the type a+ib, where a and b are real numbers, and i is one of the square roots of (-1)..." (p. 5). Results of their study showed that through questioning, even when leading, teachers' encouragement of students for thoughtful and investigative thinking might yield a shift in the discourse from real to complex numbers both algebraically and geometrically. Furthermore, Panaoura et al. (2006) found that secondary school students viewed complex numbers' algebraic and geometric representations distinctly and not as two means of representing the same number. They also claimed that this result might be due to the fact that complex numbers are introduced "…usually with little or no visual or geometric interpretation" (Panaoura et al. 2006, p. 684).

Research studies with secondary teachers (Conner et al., 2007; Karakok et al., 2015) also pointed to teachers' difficulties in conceptualization of complex numbers. Specifically, working with three secondary school mathematics teachers on their connections of Cartesian, polar, and exponential forms of complex numbers, Karakok et al. (2015) showed that one teacher had difficulty visualizing complex numbers as points on the complex plane. Similarly, once she needed to represent i as a point, "..she was hesitant whether it was located one unit up from the origin…" (p. 339). Also, another in-service teacher stated "Well, I guess just thjs letter i and anything that correlates with having i so like i+1 and multiplies of I and all that" (p. 340). So, the researchers concluded that the teacher seemed to relate the cartesian form of a complex number as an algebraic process performed on i. Similarly, two teachers in the study "…both had difficulties relating ….vector

representations of the Cartesian form" (p. 345). In addition, prospective teachers did not reconnect complex numbers to the roots of quadratic equations (Conner et al., 2007).

Moreover, Nemirovsky et al. (2012) study with prospective secondary teachers showed that the classroom floor setting offered to discover the geometric meaning of addition and multiplication of complex numbers afforded prospective teachers to consider multiplying with i as a 90-degree rotation. Building on these studies, Saraç (2016) studied a prospective teacher's development of the Cartesian form of complex numbers through quantitative reasoning. Results have shown that the prospective teacher was able to conceptualize complex numbers as a single entity, as an element of a well-defined set. Particularly, she conceptualized complex numbers as the set of the roots of any quadratic equation with real coefficients. Similarly, given any complex number, a+bi, she was able to explain what a and b represented both algebraically and geometrically. She also was able to explain why and how the conjugate roots exist given a complex root of any quadratic equation. In addition, contrary to Karakok et.al. (2015) results, she was able to locate i as referring to a point, (0,1), on the complex plane (Saraç & Karagoz Akar, 2017).

All these studies point out that in-service teachers and students need to have a complete understanding of complex numbers, including the Cartesian form with a focus on algebraic and geometric meaning of the unit i. In this study, we propose that while developing teachers' conceptions of complex numbers, quantitative reasoning might provide a robust thinking process for teachers.

Thompson (1990) described quantitative reasoning as an individual's "analysis of a situation into a quantitative structure" (p. 13). From the perspective of Thompson (1994), "quantities are conceptual entities" and "a person is thinking of a quantity when he or she conceives a quality of an object in such a way that this conception entails the quality's measurability" (p. 184). We view complex numbers as the union of two quantities as directed distances derived from the analysis of a situation (a mathematical object such as quadratic functions) into a network of quantities, which are the roots and the abscissa of the vertex as distances from the origin and quantitative relationships (See for a more detailed description, Karagöz et al., in press). By quantitative relationships, we refer to the relationship between the roots and abscissa of the vertex position on the number line as a point and as a distance. It is in this regard that we scrutinized the following research question:

> How do in-service teachers conceptualize i algebraically and geometrically upon completion of a PD program focusing on quantitative reasoning?

Method

This study was part of a Teacher Development Experiment study (TDE), assuming a design-based research (DBR) approach. For the advancement of the content knowledge of complex numbers, we focused on teachers' professional development (PD). TDEs include both classroom teaching experiments and (multi) case studies

(Simon, 2000). In this paper, we presented the findings from the multi-case study focusing on teachers' existing knowledge base, upon completion of the PD. Four teaching sessions lasting between 120 and 150 minutes made up the PD. The first two sessions were about the Cartesian form of complex numbers with quadratic equations through quantitative reasoning (Saraç & Karagoz Akar, 2017), the third one on the polar form, and the last on the Euler form. As the nature of DBR investigations, the PD focused on the theory of quantitative reasoning by considering complex numbers in a quantitative structure as a union of two quantities, which are directed distances and the roots of any quadratic equation with real coefficients.

Five mathematics teachers, with two to ten years of teaching experience and a degree in secondary school mathematics teaching, participated in the study. Initially, ten teachers completed a pre-written session on complex numbers. They were asked to define quadratic functions and equations, different forms of complex numbers, and vectors algebraically and geometrically. Eight teachers were purposely selected based on the preliminary analysis of the pre-written session and on the following criteria: They knew quadratic functions and defined and expressed vectors, which were considered background knowledge. However, they did or did not state complex numbers in the cartesian, polar, and/or Euler forms and did not explain the relationship between different forms, considering the pedagogical goals (Simon, 2000). Five of these eight teachers declared their availability to attend the study.

A post-written session was held after PD was finished. Then, the researcher, who also implemented the PD to collect data about how the participants conceptualized the connections among different forms of complex numbers, carried out video-recorded semi-structured interviews. Interviews lasted between 30 min. to 45 min. Participants' written artifacts were also gathered.

For data analysis, the constant comparative method (Clement, 2000) was used. The research team collectively read the participants' written responses and transcripts of all the interviews, watching the video recordings, when necessary, in order to characterize how teachers conceptualize the complex number i algebraically and geometrically. The participants' statements in the transcripts ranging from a sentence to a whole paragraph on each interview question served as the unit of analysis. We conducted the analysis utilizing the constructs of quantitative reasoning. The analysis focused on how the teachers were making sense of the complex number i quantitatively and explaining the meaning of i as dilation and rotation operators quantitatively. We analyzed the data first for each participant and then for each question from different participants. We created narratives describing how participants conceptualized complex number i through quantitative reasoning, by comparing and extracting the similarities and differences between their thought processes.

Results

In the following sections, the results related to different conceptualizations of i such as the definition, and geometric representation, of i will be explained. Moreover, the operator meaning of i will be discussed with

examples from interview data.

Different Conceptualizations of i

All the participants could define i in the post-written session. Among them, T2, T4 and T5, and T1 stated that i is one of the roots of the quadratic equation, $x^2 + 1 = 0$. In their explanations, they further stated that they take $\sqrt{-1}$ as equal to i. Only T3, defined that "i is the number which places at the point (0,1) in the imaginary axis". Similarly, all the participants but T4 could show i as a point on the complex plane as (0,1) geometrically. Rather she wrote (0, $\sqrt{-1}$) by stating that this point refers to the $\sqrt{-1}$ unit distance on the imaginary axis. We now provide data on T4' reasoning about showing i on the complex plane during the post- interview. T4 again defined i as one of the roots of the quadratic equation $x^2 + 1 = 0$. She also stated that i is the number whose square is equal to -1 and further acknowledged i as equal to $\sqrt{-1}$. Though, as her drawing showed, T4 first showed i by writing (0, $\sqrt{-\Delta}/2a$) and then by writing i and providing an example she also wrote $(3i)$ on the imaginary axis (See Figure 1):

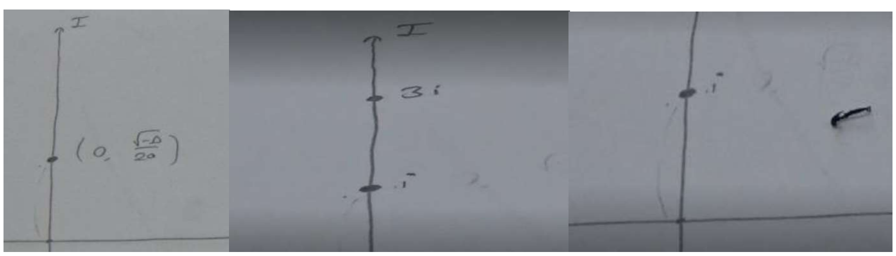


Figure 1. T4's drawing on geometric representation of i

Then hesitantly she stated whether she should use (0,3) or not. She also mentioned that i was 1 unit above the origin on the imaginary axis. Discussion continued:

> R: How do you know it's 1 unit?
> T4: We said square… square of it is -1, its distance is 1.
> R: What does "its square is -1" mean, so how do you know it's 1 unit?
> T4: Actually, for example, when I rotate this 90° (referring to what she drew on the y-axis), let this be -1. When I rotate this 90°, I am here again, -1 (marking the distance of 1 unit on the - y on the axis) becomes 1 (indicating the rotation sign again and marking a distance of 1 unit on the x-axis). Here I am actually tracing a 1unit circle.

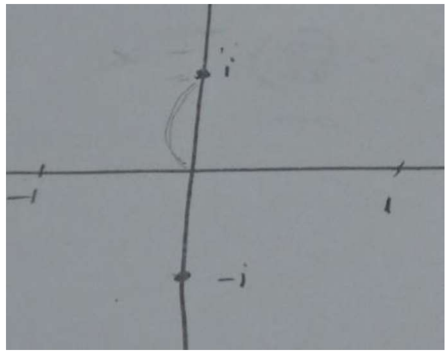


Figure 2. T4's drawing on multiplication of i

R: How do you know?

T4: We mentioned it while talking about polar representation. For example, its length, if I wanted to write this, it would be x+iy as x is 0, y in iy is 1.

R: What is y? Can you show me here where y is?

T4: This part is $(\sqrt{4ac-b^2})/2a$, when I write this as x+iy, here (referring to x) is -b/2a here (referring to y) is $(\sqrt{4ac-b^2})/2a$ so if I say i here (writing (0,i1) it becomes 1 unit.

R: You say $(\sqrt{4ac-b^2})/2a$ would be 1 unit.

T4: Huh, yes, it has become 1 unit.

R: Then what happens if you show as a point?

T4: I can't, I don't know, I can't be sure about that (0,1).

R: Why? You weren't sure about (0,3) either, were you?

T4: Yes, I wasn't sure about (0,3) either because we talked about something, for example, we said that we can say 3i and i as distance, for example, we couldn't compare them in terms of size. Yes this is 1 unit of distance (referring to the distance between i and the origin) this is 3 units of distance (referring to the distance between 3i and the origin) but when I show it as a point, I am not sure if I can write (0,3) like a coordinate.

R: And is there a contradiction between what you said and this.

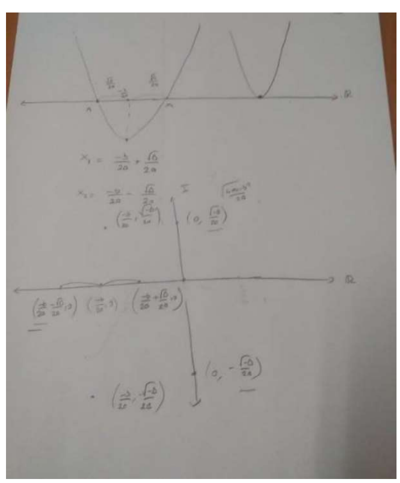


Figure 3. T4's drawing on the roots and the vertex

T4: Yes, then I can write.

R: I don't know which one you say is correct? Which one shall I believe?

T4: Here (referring to √(-Δ)/2a) is (√(4ac-b²))/2a anyway

R: What was that geometrically?

T4: It was y. Then I can write (0,1), (0,3) OK.

R: Tell me again what has changed, why has it changed?

T4: I could not, we couldn't compare the size of 3i and i there. You know, as a point what we wrote in the second part is a point on the complex plane, but now it's ok.

R: What do you mean, what's okay?

T4: When I write the coordinates there, I already use real numbers. When I say x+iy there, I can write the y part (0,3) there. I just had wondered if I could write whether it is (0, 3i) but it is (0,3) yes. Now I can write this (referring to (0,3)). Let me write then (putting the points back). This is 3, this is 1, but which one of them is bigger, I can't say.

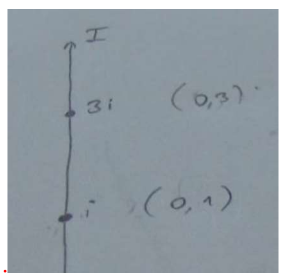


Figure 4. T4's drawing on i and $3i$

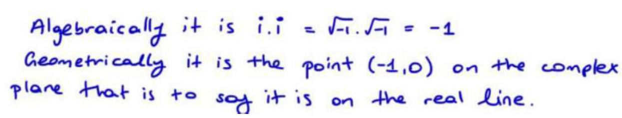
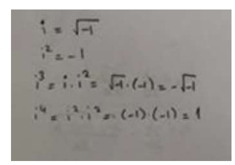
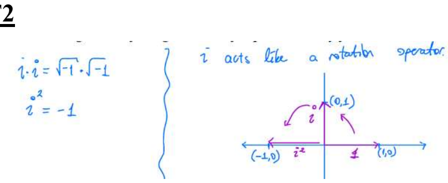
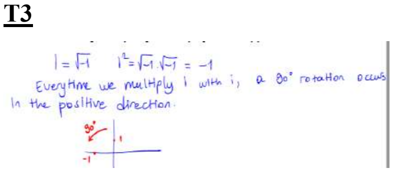
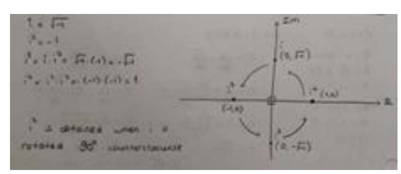
As the data indicated, T4 had difficulty in showing i as a point (0,1) on the imaginary axis. Notably, she knew that i was 1 unit above from the origin. She also knew that $3i$ was 3 units away on the imaginary axis from the origin. She knew this because as she stated, i was on the unit circle where she considered i as an operator of 90-degrees rotation counter clockwise (See Figure 2). Yet, she knew from the discussion during PD that complex numbers could not be ordered and this made her think that it might not be possible to represent i as (0,1). That is, as if when she used (0,1) or (0,3) for representing i and $3i$ respectively, she would revert back to real numbers, which could be ordered (i.e., 1 is smaller than 3). More specifically, she considered that if she had used (0,1) as could be written on R^2, this would contradict the information that the complex numbers could not be ordered. Therefore, she did not know whether she could write (0,3) to indicate $3i$. She seemed to be considering that points on the imaginary axis should be shown by imaginary symbols. However, when probed to think about her earlier drawing showing the roots on the complex plane, she recalled that complex numbers are represented by ordered pairs of real numbers. This then allowed her to show i as (0,1). These data suggested that, for T4, the polar form of complex numbers was more meaningful than the binomial form. Another important observation is that T4's explanations about comparing real and complex numbers in terms of ordering property are not legitimate when R^2 and the complex plane is considered. That is, in R^2 there is no notion of

ordering either. This suggests that T4 might not have known that there is one-to-one correspondence between the ordered pairs in R^2 and complex numbers.

Meaning of i as dilation and rotation operators

As the data in Table 2 indicated, in the post-written test all the participants could explain the powers of i both algebraically and geometrically as 90 degrees rotation counterclockwise. Yet, during the interview, T1's explanations pointed to complex number bias while T5's explanations were valid. Also, T1 could mention not only the operator but also the dilation meanings of multiplication of complex numbers.

Table 2. Teachers' post test answers on the meaning of i

Multiplication with i	Teachers	Sample data
Explaining powers of i using algebraic meaning	T1, T2, T3, T4, T5	**T1**  **T2** **T3** **T4**   **T5** 
Explaining powers of i using operator meaning	T1, T2, T3, T4	**T2**  **T4** **T3**  

We now provide data from the interviews with T5 and T1, when asked to explain how they interpreted i^2:

T5: Because i was equal to $\sqrt{-1}$. Therefore, the number we called i^2 became $\sqrt{-1}.\sqrt{-1}$. This gave us

the number -1. -1 is a real number. So if we write this as a complex number, in the format $x + iy$, it becomes 0+, sorry, $-1 + 0i$. That's why I said we can show it as a number, as a point on the x-axis, at a distance of 1 unit form 0, on the left side, the negative side. So, I said directly it will be on the real axis.

Comparatively, data from T1 pointed to complex number bias:

T1: Let's write i squared (i^2) algebraically. Algebraically, it is multiplying i by i, root minus one by root minus one ($\sqrt{-1}.\sqrt{-1}$). Our rules in real numbers are valid here. We can multiply two numbers inside a single root. It becomes root one ($\sqrt{-1}$). So it's one. Um, where do we put it? The result is a real number, so we multiplied two imaginary things and the answer came out real. Then I'll show on the real.

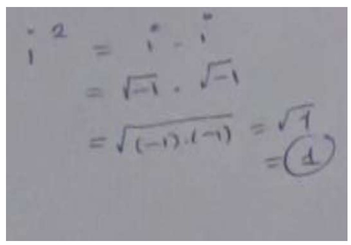


Figure 5. T1's explanation on i^2

That's here somewhere, I'm trying to make these things equal. i squared will be at (1.0). Isn't it? Yes. Because i squared is positive. I mean, I think that we cannot write i squared (i^2) under it, because it is i squared. One of its points is (1,0). One minute, i squared would be minus one, right? Smoke will come out of my head soon. One minute. I multiplied i with i, that is i squared. The i squared has to be minus one. Sorry, it won't be like this. This will be minus one.

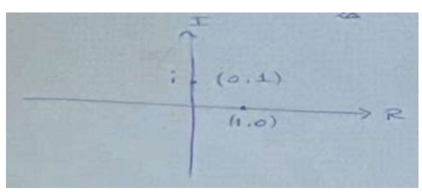


Figure 6. T1's drawing on the place of i

R: Why, how?

T1: Because I can't call it minus one squared, minus one squared $(-1)^2$). I'm saying it wrong. How can I explain that? I'm making this up now, if it was root two ($\sqrt{-2}$) and root two ($\sqrt{-2}$), it would be two. It will also be negative one since it is root minus [times] one root minus one ($\sqrt{-1}.\sqrt{-1}$). So I'm trying to explain it with that logic. Root will go. From the same number. If there are two identical numbers in

the square root, that number comes out when we multiply them. This is minus one, not here, on this side (she shows it in the complex plane.) Again, it's real.

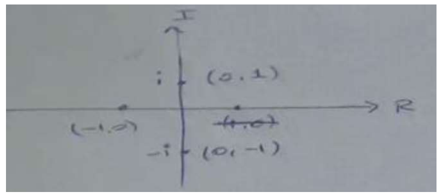


Figure 7. T1's drawing on the place of $-i$

i cubed (i3) is minus one multiplied by i, minus i, that's over there. What will it become? It becomes (0,-1). Just below this (shows the point (0,1)). Yes.

R: Okay. Can you tell me again why you changed your mind here?

T1: I changed my mind here because of this: This is not a real thing [number]. i is not a negative number. i is something like root minus one.

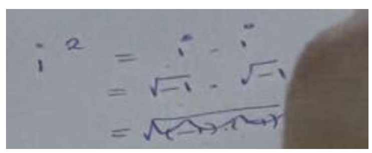


Figure 8. T1's algebraic explanation on i^2

T1: That's why I changed my mind. Because if i was the number -1, the square of -1 will come out as 1 anyway. There is no problem there. But this is defined as root -1 anyway, there is no such thing normally. But there are two expressions inside the root. Both are the same. It is like, as if the roots have canceled out each other. So it's like it turned into a square. That's why I changed my mind. 'Cause if I do this it'll be like I'm squaring -1.

T1: Then it will be as if I have defined it like this.  But it is not like that 

As the data indicated, T5 could verify that $\sqrt{-1}.\sqrt{-1}$ is equal to-1. Also, she could explain how she could write -1 as (-1,0) by pointing to the binomial form of complex numbers such that she matched $x + iy$ with $-1 + 0i$ as (-1,0). Yet, she did not explain how she knew or how she could deduce $\sqrt{-1}.\sqrt{-1}$=-1 and the researcher did not probe her further to explain her reasoning. T5's earlier explanations regarding that i is the number whose square is -1 might be the reason to write such equality. On the other hand, T1's explanations suggested that T1 considered the properties of the radicand sign in Real numbers and the complex numbers the same. Her earlier explanation and writing (See Figure 8) showed that she considered that the rule $\sqrt{a}.\sqrt{b} = \sqrt{a.b}$ in real

numbers also holds for complex numbers. This pointed to a complex number bias. However, similar to all other participants, when T1 considered that i is the number whose square is -1 she realized that she made a mistake. Her explanations suggested that she considered $\sqrt{a}.\sqrt{a} = a$ also held for $a = -1$.

Before talking about i^2, T1 mentioned multiplying i with $\sqrt{-\Delta}/2a$ so the researcher asked how she made sense of it.

R: This i times $\sqrt{-\Delta}/2a$

T1: Where will it be?

R: What does it correspond to, yeah, where will that number be?

T1: On here (showing imaginary axis). It is on the imaginary axis. It depends on the value of the coefficient. So for instance, if $\sqrt{-\Delta}/2a$ is 3, I'll be pointing at $3i$. Or like I'll be pointing at $-3i$.

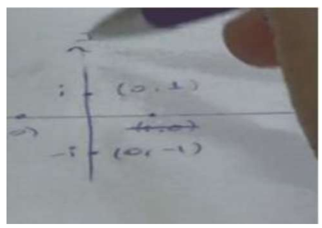


Figure 9. T1's drawing on i and $-i$

It's not a real number, but it has a real coefficient. I can say it is getting bigger or smaller according to that. Saying getting bigger or smaller may not be very accurate, as I said, because of the definition of i, it may not be like is i bigger than 3i, but we can assume that the distance is the distance to zero and place it (talking about real number times i) Since that is positive, we can place it.

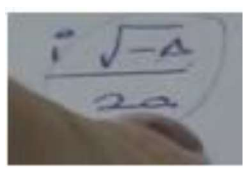
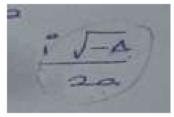


Figure 10. T1's written work on multiplication with i

R: OK. You're telling me… you say that when I multiply it with i, I can place it here (pointing to theimaginary axis). I'm asking you if you can explain √(-∆)/2a when multiplied with i.

T1: I could not perceive the difference exactly, but I will. I multiply from here, you multiply from here (meaning multiplying from the right or left side)… and it makes a difference. Okay, just a minute. Umm, does this still exist? This (-b/2a) is zero and I'm saying this (i.(√(-∆)/2a))…Okay. Ok. I need to put it somewhere on the axis. Because these are the i (pointing to the imaginary axis.) This (pointing to i.(√(-∆)/2a)) I'll call it a certain coefficient of i. Because it's a coefficient.



R: OK. Alright. I am asking the same question again. What kind of number is √(-Δ)/2a? What does it mean when you multiply a real number with i? You tell me what it means when you multiply i with a real number.

T1: OK, one minute. What was it? Do I explain what happens when I multiply i with a real number? One minute. Ok. Ok. What happens when I multiply i with a real number? OK, when we multiply a real number with i, that number becomes an imaginary number…

R: I'm asking what it means. It's imaginary, on that we agree.

T1: okay, for example √(-Δ)/2a was on this line (referring to the real axis). Because it was real. When I multiply it with i, shall I say if it rotates, what shall I say? It went this way. It moved to the imaginary axis. It rotated, it rotated. It rotated, from here (meaning the real axis) to here (meaning the imaginary axis). Rotated ninety degrees, yes.

Importantly, the data showed that T1 initially interpreted the question $\sqrt{-1}\sqrt{-\Delta}/2a$ as the multiplication of i with a real number, referring to scalar multiplication rather than interpreting it as a real number multiplied with i, as rotation operator. First of all, syntactically this might make sense as the order in which the multiplication is symbolically written might have triggered this meaning. Still, T1 seemed to think that a real number acted on the $\sqrt{-1}$ which made it longer or shorter. This suggested that T1 made sense of the multiplication of i with a real number as a dilation operator. That is, she seemed to think of $\sqrt{-1}$ as the multiplicand and the $\sqrt{-\Delta}/2a$ as multiplier. This was probably possible for two reasons: First, T1 made sense of the binomial form of complex numbers as vectors. As the figure showed, she even mentioned i as a unit vector, which refers to (0,1) and she made sense of $\sqrt{-\Delta}/2a$ as the distance to the origin. Considering the location of i, she even located $3i$ three units away from the origin on the imaginary axis. Secondly, her referring to $(-b/2a)$ as being zero and her statement that $\sqrt{-\Delta}/2a$ is a real number suggested that she made sense of the dilation operator referring back to complex numbers as the roots of quadratic equations. However, as the data indicated, T1 had difficulty in thinking with i as a rotation operator. Only after the researcher has taken her attention three times, T1 could attend to the difference between i as a multiplier versus i as a multiplicand. That is, only after being prompted, she stated that i as a rotation operator, the multiplier, acts on the $\sqrt{-\Delta}/2a$, rotating it 90 degrees and locating any real number on the imaginary axis. All these data showed that T1 could reason on multiplication of i with real numbers in two ways albeit some difficulty.

Conclusion and Discussion

Results of the study showed that, upon completion of a PD study focusing on the conceptualization of different

forms of complex numbers, in-service teachers developed a meaning of i both algebraically and geometrically. In addition, although all of them could mention the rotation operator meaning of i, one of them was able to consider ib pointing to both i acting on b as a 90-degrees rotation and b acting on i as a dilation operator.

Particularly, in accordance with researchers' emphasis about the conceptualization of i (Kontorovich, 2018b; Nachlieli & Elbaum-Cohen, 2021), results showed that four out of five inservice teachers defined i as one of the roots of the quadratic equation, x^2+1=0 and one teacher did not mention this. However, they also stated that they accepted √(-1) as equal to i. Such conceptualization is valid as in formal mathematics i is considered as the principal root of √(-1) (Usiskin et al., 2003). In addition, T1 specifically showed on the complex plane that she also had an awareness of "-i" as one of the roots of x^2+1=0. Still, complying with Nachlieli & Elbaum-Cohen (2021) suggestion, we propose researchers and teachers, while both teaching and doing research on different forms of complex numbers, to include a discussion about the importance of taking into consideration of i as one of the square roots of √(-1), the principal root. This might be important for both students and teachers to pay attention to the polysemy of the radical sign (Kontorovich, 2018b). This might be further important as the results showed, contrary to one teacher, T5, at first another teacher, T1, considered that the rule √a.√b=√(a.b) in real numbers also holds for complex numbers, indicating complex number bias (Kontorovich, 2018a). However, whenever she thought of i as the number whose square is -1, she could overcome her difficulty with complex number bias on her own.

Another important observation is that T4's explanations about comparing real and complex numbers in terms or ordering property is not legitimate when considering R2 and complex plane. T4 had difficulty in positioning i on the complex plane as she considered order quality within the realm of real numbers. However, in R2 ordering the elements is not possible either. This suggests that T4 might not have known that for any number pair on R2, there exists exactly one pair of numbers (a,b) on the complex plane such that there is one-to-one correspondence between the ordered pairs in R2 and complex numbers, an isomorphic structure (Kontorovich et al., 2021). Thus, we argue that both researchers and teachers might provide opportunities with learners to first establish R2 quantitatively on the part of both students and teachers (Karagoz et al., 2022). Previous research has pointed to the importance of understanding of a point as a multiplicative object (Karagoz et al., 2022; Thompson et al., 2017) that is formed from two quantities by mentally uniting "their attributes to make a new attribute that is, simultaneously, one and the other" (Thompson et al., 2017, p.96). Karagoz et al. (2022) further stated that thinking about a point on the plane, an ordered pair in the form of (a, b), as a multiplicative object as the point say, A = (a, b) can be thought of as a cognitively uniting of two quantities' magnitudes (Saldanha & Thompson, 1998; Stevens & Moore, 2017). This also aligns with Gravemeijer (2020)'s emphasis that learners should understand that each point on the Cartesian plane is a number pair signifying two connected measures. As the results of the study suggest, conceptualizing R2 in a quantitative structure might be a precursor to an understanding of isomorphism between R2 and the complex plane.

Furthermore, aligned with earlier research (Nemirovsky et al., 2012; Soto-Johnson and Troup, 2014), results showed that all the teachers could think of i as a 90-degrees rotation operator. Results also showed that this

further allowed for all the teachers to think of the powers of i as points located on the complex plane. Moreover, one teacher could think of ib considering both the rotation and dilation operator meanings. That is, results suggested that she could consider i as a multiplier and b as a multiplicand and b as a multiplier and i as a multiplicand. We argue that such a dual understanding of ib needs to be further investigated. Previous research has pointed to how a classroom floor afforded the ways for pre-service teachers to make sense of multiplying with i as a rotation operator. However, how learners come to such an understanding might be further investigated with clinical studies, such as teaching experiments. In addition, as the results pointed, such understanding is one part of the coin, which also includes thinking of b as a multiplier and i as a multiplicand. Research on multiplication with rational numbers points to two types of reasoning models, namely as repeated addition and as multiplicatively. The repeated addition view is considered as limited and primitive (Fischbein et al., 1985). Similarly, Thompson and Saldanha (2003) argued that "repeated addition is a quantification technique; it is not the thing being quantified" (p. 104). On the other hand, understanding of multiplication multiplicatively requires thinking about "times as much" (Thompson & Saldanha, 2003) as the learners consider "...both the product and the factors of the product (i.e., the multiplier and the multiplicand) in relation to it" (Karagöz et al., 2022b, p.114). Further thinking about the dilation (i.e., stretching and shrinking) meaning of multiplication, we argue for extending research on multiplication with rational numbers, and doing research on understanding multiplication of i with real numbers. This is specifically important as the results in T1's explanations suggested that a dual meaning of multiplication of i with real numbers might have roots in understanding the roots of quadratic equations quantitatively in relation with vectors. Also, previous research has shown that pre-service teachers do not have a multiplicative understanding of multiplication (e.g., Karyağdı, 2022).

In this study, focused on five inservice teachers. Further research can be done to investigate students' and preservice teachers' conceptualizations of different meanings of i with a larger number of participants.

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